# Minimum Model Error Estimation for Poorly Modeled Dynamic Systems

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**A novel strategy (which we call "minimum model error" estimation) for postexperiment optimal state estimation of discretely measured dynamic systems is developed and illustrated for a simple example. The method is especially appropriate for postexperiment estimation of dynamic systems whose presumed state governing equations are known to contain, or are suspected of containing, errors. The new method accounts for errors in the system dynamic model equations in a rigorous manner. Specifically, the dynamic model error terms in the proposed method do not require the usual Kalman filter-smoother process noise assumptions of zero-mean, symmetrically distributed random disturbances, nor do they require representation by assumed parameterized time series (such as Fourier series). Instead, the dynamic model error terms require no prior assumptions other than piecewise continuity. Estimates of the state histories, as well as the dynamic model errors, are obtained as part of the solution of a two-point boundary value problem. The state estimates are continuous and optimal in a global sense, yet the algorithm processes the measurements sequentially. The example demonstrates the method and shows it to be quite accurate for state estimation of a poorly modeled dynamic system.**

## Introduction

A LARGE number of applications exist in the general area of "postexperiment" estimation, wherein estimates of the actual state histories of a dynamic system are obtained using an assumed state dynamic model and sets of discrete measurements. Applications are found throughout engineering, but are especially numerous in such aerospace problems as orbit estimation, attitude estimation, and postflight trajectory estimation.

In general, both the dynamic model and the available measurements are imperfect. The motivation for applying an estimation algorithm is to combine the model-predicted state estimates with the available measurements in such a way as to obtain estimates of the state histories which are of higher accuracy and more complete than either the model predictions or the measurements. If the estimation algorithm optimizes some performance index, typically based on the state estimate error, then the resulting state estimate is said to be optimal. In this paper, we propose a novel optimal estimation strategy, which includes both a new optimality criterion and a new algorithm for obtaining estimates based upon it.

The most commonly used estimation approach is the Kalman filter and numerous closely related strategies, originally developed by Kalman[1] and Kalman and Bucy.[2] In the keynote address to the 1985 American Control Conference, Gelb[3] points out that 2997 papers on Kalman filtering were published in the 15 years between 1969 and 1984, an average of 200 each year or 17 each month. The Kalman algorithms are well-suited for real-time estimation, because of their sequential processing structure and emphasis on the most recent data. Probably as a result of popularity and familiarity, Kalman filters are now routinely used for postexperiment estimation, normally accomplished via iterative postprocessing of the filter estimate ("smoothing"; e.g., Gelb[4]).

The various filter strategies use similar, although distinct, optimality criteria. The two most common are "minimum variance" and "maximum likelihood" criteria (see, e.g., Junkins[5]). In minimum variance estimation, a function of the trace (typically the trace itself) of the state estimate error covariance matrix is minimized. In maximum likelihood estimation, the most probable state estimate is found given the measurements. The important feature of these existing strategies is that they require estimation of the state estimate error covariance. In order to rigorously estimate the state estimate error, knowledge of both the model error and the measurement error is required. While measurement errors may be determined in numerous ways, model errors are generally unknown by definition (i.e., if one is aware of the model error, one corrects the model).

To the best of the current authors' knowledge (we admit that we have not read all of the pre-1969 nor post-1984 Kalman filter papers, let alone the 2997 published during 1969-1984), every filter strategy actually implemented numerically deals with the model error knowledge requirement by assuming that the model error is a symmetrically distributed white noise sequence of known covariance, normally called "process noise" in the literature. We note that this assumption often has no theoretical basis, and that, in fact, for physical systems, model errors are more often smooth functions resulting from typical model simplification assumptions such as linearization, ignoring of secondary effects or higher-order terms, etc., or just plain ignorance.

In this paper, a new optimality criterion for determining state estimates is described. This criterion seeks to obtain the smallest estimate of model error subject to a "covariance constraint." It is based solely on assumed knowledge of the measurement error covariance, which, we claim, is far more likely to be known accurately than the model error and which, in any event, is also required in the filter strategies. We also present an optimal strategy ("minimum model error" estimation) for determining the state estimates which satisfy the optimality criterion. The algorithm is shown to be related to classical optimal control problems.

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## General Problem Statement

The following generic problem statement for post-experiment estimation of a dynamic process is used as a starting point for the development of the method. Given a system whose state vector dynamics is modeled by the (linear or nonlinear) system of equations,

$$\dot{x}=f[x(t),t] \tag{1}$$

where

$x=n\times 1$ state vector

$f=n\times 1$ model equations

and given a set of discrete state-observable measurements modeled by the (linear or nonlinear) system of equations,

$$\tilde{y}_k=g_k[x(t_k),t_k]+v_k, \qquad k=1,2,\ldots,M \tag{2}$$

where

$\tilde{y}_k=m\times 1$ measurement set at time $t_k$

$g_k=m\times 1$ measurement model at time $t_k$

$v_k=m\times 1$ gaussian distributed random sequence, with zero mean and known covariance, $R_k$

determine the optimal estimate for $x(t)$ [denoted by $\hat{x}(t)$], during some specified time interval $t_0\le t\le t_f$.

## The Covariance Constraint Concept

In the present method, the optimal state trajectory estimate is determined on the basis of the assumption that consistent estimates of the state trajectories must match the available measurements with a residual error covariance which is approximately equal to the known measurement error covariance. This necessary condition is hereafter referred to as the "covariance constraint." The covariance constraint is imposed by requiring the following approximation to be satisfied:

$$\{\tilde{y}_k-g_k[\hat{x}(t_k),t_k]\}\{\tilde{y}_k-g_k[\hat{x}(t_k),t_k]\}^T\approx R_k, \qquad k=1,2,\ldots,M \tag{3}$$

Thus, the estimated output $g_k(\hat{x}(t_k),t_k)$ is required to fit the actual measurements $\tilde{y}_k$ with approximately the same error covariance as the actual measurements fit the truth. Otherwise, the estimate is statistically inconsistent.

Considerable flexibility exists in the application of the covariance constraint, Eq. (3). The interpretation of the "approximately equal" sign in Eq. (3) may be taken in a number of ways, no one of which is appropriate for all situations. For example, in the typical case where the measurement sets are all of the same physical quantities, and with the same nominal accuracy, it is appropriate to compare the averaged measurement-minus-estimate residual error covariance with a single prescribed measurement-minus-truth error covariance as

$$\frac{1}{M}\sum_{k=1}^{M}\{\tilde{y}_k-g_k[\hat{x}(t_k),t_k]\}\{\tilde{y}_k-g_k[\hat{x}(t_k),t_k]\}^T\approx R \tag{4}$$

where the subscript $k$ has been dropped for the prescribed covariance to indicate that the measurement-minus-truth covariance is constant over all measurement sets.

The averaging of covariances over a number of measurement sets is applicable if any of the measurement sets are repeated during the time interval of interest. In some applications, there may be two or more distinct measurement subsets, each of which is measured numerous times. In these situations, an averaged covariance constraint in the form of Eq. (4) may be applied to each distinct measurement subset. An example of this type of application is the attitude estimation of a spacecraft, where numerous angular velocity and independent attitude angle measurements are made, but at different times and with different accuracy. An averaged covariance constraint for angular velocity measurements and a separately averaged covariance constraint for attitude angle measurements is appropriate.

The primary motivation for using an averaged covariance constraint is the reduction of the sensitivity of the approach to small sample statistical anomalies. If the covariance constraint is imposed individually on each measurement set, then obviously any measurements which deviate significantly from the assumed measurement error may cause unrealistic corrections to the state estimates. By averaging over a large number of measurements, the likelihood that statistical anomalies are affecting the estimates is reduced. We note in passing that the usual filter-smoother measurement processing equations are applied to each measurement sequentially, with the assumption that the measurement error covariance for that measurement is correct. Thus, considering small sample statistics sensitivity problems, the worst case in the present approach (applying a covariance constraint to each measurement set) corresponds to the standard filter approach (calculating gains at each measurement set based on assumed sample covariances).

The estimate optimized by the averaged covariance constraint is in essence a global best fit, i.e., a batch estimate. In general, if a number of measurement sets are available simultaneously, as is the case in postexperiment estimation, batch processing is preferred over sequential filtering due to smoother estimates and generally higher accuracy. The advantages of sequential filtering are found in real-time estimation, which is not the case under study here, and in the ease of computation which is gained by processing the measurements sequentially. As is shown in a later section, we may accomplish a batch estimate via sequential processing of the measurements.

## The Minimum Model Error Concept

We begin by accounting for model errors by adding a to-be-determined unmodeled disturbance vector $d(t)$ to the right-hand sides of the original state model equations, Eq. (1), to produce the modified state governing equations,

$$\dot{x}=f[(x(t),t)+d(t)] \tag{5}$$

Next, the following cost functional is minimized with respect to $d(t)$:

$$J=\sum_{k=1}^{M}\{\tilde{y}_k-g_k[\hat{x}(t_k),t_k]\}^TR_k^{-1}\{\tilde{y}_k-g_k[\hat{x}(t_k),t_k]\} + \int_{t_0}^{t_f}d^T(\tau)Wd(\tau)d\tau \tag{6}$$

where

$W=n\times n$ weight matrix, determined so as to satisfy the covariance constraint as described shortly.

An algorithm for the minimization of Eq. (6) with respect to the unmodeled disturbance vector $d(t)$ is developed in the next section.

The functional $J$ in Eq. (6) is the sum of two penalty terms. The first is a weighted sum of discrete terms which penalize the deviation of the predicted measurements (based upon the output computed using the estimated states) form the actual measurements. Minimization of this summation term drives the state estimates toward values which, when substituted into the measurement model, predict the actual measurements. The weighting $R_k^{-1}$ on each of these penalty terms is the inverse of the associated measurement error covariance; thus, accurate measurements (small $R_k$) are weighted more heavily than inaccurate measurements (large $R_k$), as in weighted least squares or maximum likelihood estimators. The second term in $J$ is an integral term which reflects the assumption that the amount of unmodeled effect to be added should be minimized, i.e., the original model should be adjusted by a minimal amount (we give the modeler the benefit of the doubt. While $d(t)$ may in fact be large, it should be as small as possible!). This is the origin of the title "minimum model error" estimation.

The presence of $d(t)$ in the cost functional $J$ produces somewhat ambivalent results. If the original state model equations contain significant errors, then the addition of large model correction terms should enable the estimate to better fit the measurements. Thus, the summation term in $J$ is decreased. However, the addition of $d(t)$ increases the integral term in $J$. The proper balance between the two competing effects depends on the choice of $W$. The weight matrix, $W$, is determined such that the covariance constraint is satisfied. Thus, we seek the smallest $d(t)$ which is consistent statistically with the measurements.

A typical plot of $W$ vs the covariance constraint is shown in Fig. 1a. To keep the discussion simple, assume that a single state is being estimated, and that the measurements are of the state itself. Further assume that the dynamic model contains significant errors, i.e., the model-predicted state is substantially different from the actual state.

In Fig. 1a, the covariance constraint is represented by the point at which the measurement-minus-estimate variance is equal to the measurement-minus-truth variance (i.e., the known measurement error variance).

For large values of $W$, the model correction term $d(t)$ (the second term in $J$) is penalized heavily compared with the measurement-minus-estimate residuals (the first term in $J$). Consequently, large measurement-minus-estimate residuals are allowed. The state estimate is based primarily on the original model. The measurement-minus-estimate variance is large and essentially constant for a wide range of large $W$, since the model correction term $d(t)$ remains virtually zero. The measurement-minus-estimate variance is much larger than the measurement-minus-truth variance.

However, if $W$ is decreased, the model correction term is penalized relatively less heavily and the measurement-minus-estimate residuals are penalized relatively more heavily. For some sufficiently small $W$, the model correction term $d(t)$ becomes nonzero and begins to correct the model prediction toward agreement with the measurements. The appropriate value for $W$ is the one which allows enough model correction to cause the measurement-minus-estimate variance to match the measurement-minus-truth variance. If $W$ is decreased below this value, the estimate matches the measurements too closely and becomes too affected by the measurement noise.

In Fig. 1b, the actual estimate error variance (i.e., the truth-minus-estimate variance) is plotted vs $W$. Statistically, this variance is minimized when the covariance constraint is satisfied. If $W$ is too large, the estimate is too far from the measurements. If $W$ is too small, the estimate is too close to the measurements. When the covariance constraint is satisfied, the estimate matches the measurements with the same covariance as the truth matches the measurements. For the given model and particular sample of measurements, this is the optimal estimate, although the actual error covariance goes to zero only for an infinite set of measurements.

Figure 1 represents typical results in a conceptual manner. However, like all strategies based on statistical interpretation, these results are affected by the particular sample of measurement errors in the available measurements. Although the measurement error covariance matrix, $R_k$, is assumed to be known, it is strictly valid only for an infinite number of measurements, so, in practice, it is an approximation. Thus, satisfaction of the covariance constraint is also an approximation. We note in passing that filter-type algorithms require assumed values for both the measurement noise and the model error noise (of course, this latter assumption presupposes that the model error is indeed simply noise). The present method eliminates the requirement for model error assumptions but retains the measurement noise knowledge assumption. The above qualitative discussion motivates the formalization and generalization of the ideas. These results follow.

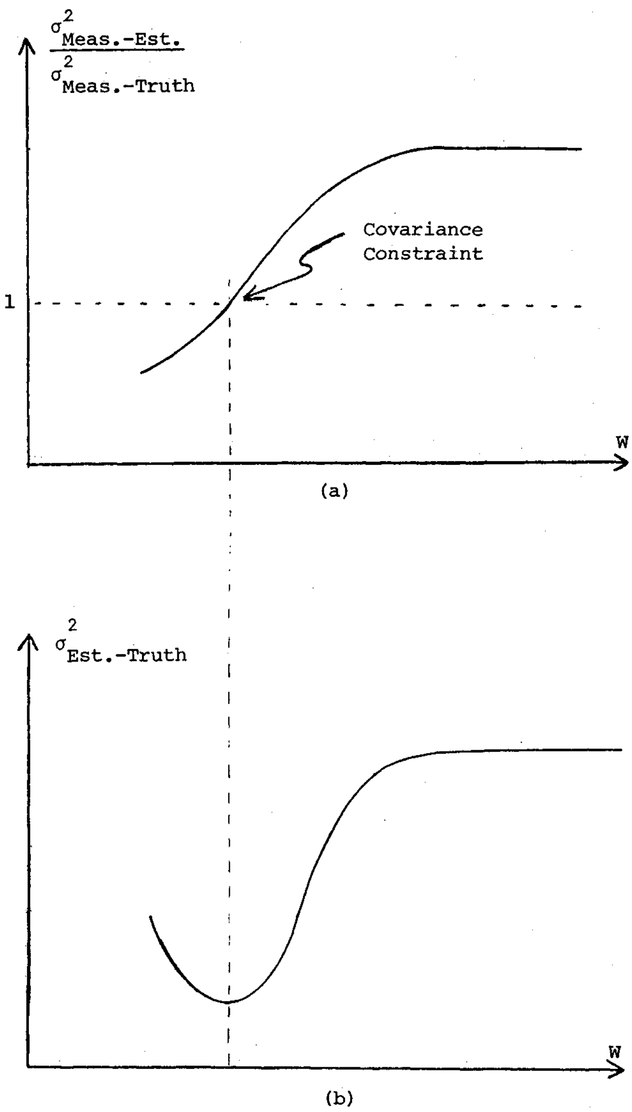


**Fig. 1 Choosing $W$ to satisfy the covariance constraint leads to an optimal state estimate.**

## Derivation of the Estimation Algorithm

In this section, the calculus of variations is used to develop an estimation algorithm for poorly modeled dynamic systems. The development is similar to the typical textbook developments of optimal control theory (see, e.g., Bryson and Ho[6] or Kirk[7]). However, the adaptation of these variational concepts to the development of an estimation algorithm represents a novel interpretation of the classical results. In particular, we encounter an unusual multipoint boundary value problem and find a novel solution process. Application of the covariance constraint is described in the next section.

We begin by giving the necessary conditions for the minimization of a functional, with respect to a vector function, which includes both discrete and integral terms. Given the system of equations,

$$\dot{x} = f(x(t), u(t), t), \qquad t_0 \le t \le t_f \tag{7}$$

where

$x = n \times 1$ state vector, $x(t_0)$ specified

$f = n \times 1$ model equations

$u = p \times 1$ to-be-determined vector for minimizations

and a performance index defined as

$$J = \phi[x(t_f)] + \int_{t_0}^{t_f} L[x(\tau), u(\tau), \tau]\, d\tau \tag{8}$$

where

$\phi$ = penalty on the final state vector

$L$ = penalty reflecting the deviation of $x(t)$ or $u(t)$ from their desired trajectories,

the problem may be stated as: find a smooth, differentiable, unbounded $u(t)$, which minimizes $J$, subject to the differential equation constraint, Eq. (7).

The necessary conditions for the minimization of $J$ in Eq. (8) are obtained by equating the variation of $J$ to zero. These conditions, usually called "Pontryagin's necessary conditions," are given by (e.g., Rozonoer[8] or Kopp[9]) the $2n$ differential equations,

$$\dot{x} = f[x(t), u(t), t] \tag{9}$$

$$\dot{\lambda} = -\left[\frac{\partial f}{\partial x}\right]^T \lambda(t) - \left[\frac{\partial L}{\partial x}\right]^T \tag{10}$$

where $\lambda(t)$ is the vector of costates: the $p$ algebraic equations,

$$\left[\frac{\partial L}{\partial u}\right]^T = -\left[\frac{\partial f}{\partial u}\right]^T \lambda \tag{11}$$

and the $2n$ boundary conditions

$$x(t_0) = \text{specified} \tag{12}$$

$$\lambda(t_f) = \left.\frac{\partial \phi}{\partial x}\right|_{x(t_f), t_f} \tag{13}$$

The $2n$ boundary conditions are split between $n$ intitial conditions on the states and $n$ final conditions on the costates (classically called transversality conditions). Thus, Pontryagin's necessary conditions lead to a two-point boundary value problem (TPBVP). Numerous methods have been developed for the solution of TPBVP's; see, e.g., Vadali[10] or Keller.[11] Typically, the $2n$ differential equations, Eqs. (9) and (10), must be integrated forward from the specified state initial conditions, Eq. (12), using "guessed" costate initial conditions. At the final time $t_f$, the integrated values of the costates are compared with the specified costate final conditions, Eq. (13). If the agreement between the integrated and specified values is inadequate, then the costate initial conditions must be adjusted and the integration repeated. This process is repeated until the agreement between the integrated final costates and the specified final costates is sufficient. Other forward/backward iteration schemes are given in Bryson and Ho.[6]

It is possible (see, e.g., Geering[12] or Mook[13]) to extend the basic Pontrayagin's necessary conditions to account for terms in $J$ at discrete times within the time domain. If the performance index is modified to the form,

$$J = \sum_{i=1}^{I} K_i(x(t_i), t_i) + \int_{t_0}^{t_f} L[x(\tau), u(\tau), \tau]\, d\tau \tag{14}$$

then the necessary conditions are modified by the additional internal boundary conditions

$$\lambda(t_i^+) = \lambda(t_i^-) - \left.\frac{\partial K_i}{\partial x}\right|_{x(t_i), t_i} \tag{15}$$

The resulting TPBVP now contains jump discontinuities in the costate at the times $t_i$ associated with each internal penalty term. These jump discontinuities complicate the calculation of the partial derivatives of the known final conditions with respect to the unknown initial conditions; these partial derivatives are frequently used as the basis for choosing corrections to the initial condition estimates. However, the solution of the TPBVP is not the focus of the present work, so the TPBVP is assumed to be solvable in the discussion which follows.

Equation (6) is clearly in the form of Eq. (14). The discrete penalty terms in Eq. (6) are given by

$$K_k = [\tilde{y}_k - g_k(\hat{x}(t_k), t_k)]^T R_k^{-1} [\tilde{y}_k - g_k(\hat{x}(t_k), t_k)] \tag{16}$$

Utilizing Eq. (15), the jump discontinuities in the costates due to this penalty term may be written

$$\begin{aligned}\lambda(t_k^+) &= \lambda(t_k^-) - \left.\frac{\partial K_k}{\partial x}\right|_{\hat{x}(t_k), t_k} \\ &= \lambda(t_k^-) + 2H_k^T R_k^{-1} [\tilde{y}_k - g_k(\hat{x}(t_k), t_k)]\end{aligned} \tag{17}$$

where

$$H_k \equiv \left.\frac{\partial g_k}{\partial x}\right|_{\hat{x}(t_k), t_k}$$

An algorithm for the implementation of the minimum model error approach now follows directly from the modified Pontryagin's necessary conditions. For a given $W$, the minimization of $J$ in Eq. (6) with respect to $d(t)$ leads to the TPBVP summarized as

$$\dot{\hat{x}} = f[\hat{x}(t), t] + d(t) \tag{18}$$

$$\dot{\lambda} = -\left[\frac{\partial f}{\partial x}\right]^T \lambda \tag{19}$$

$$\hat{x}(t_0) = \text{specified (measured, estimated, etc.)} \tag{20}$$

$$\lambda(t_0^-) = 0 \tag{21}$$

$$\lambda(t_k^+) = \lambda(t_k^-) + 2H_k^T R_k^{-1} [\tilde{y}_k - g_k(\hat{x}(t_k), t_k)] \tag{22}$$

$$\lambda(t_f^+) = 0 \tag{23}$$

This algorithm, and the resulting state estimate, exhibits several desirable features of both batch and sequential estimation techniques. The state estimate is obtained by processing all of the available measurements, much like a batch estimator such as least squares. Thus, the estimate is optimized in a global sense. In addition, the state estimate is continuous, eliminating the state estimate jump discontinuities present in filter estimates. For many physical systems, jump discontinuities in the states are not possible; thus, jump discontinuities in the filter state estimates must be reconciled in an artful manner. Obviously, the estimate of $d(t)$ is discontinuous at each $t_i$; in essence, the discontinuities have been "pushed" up one order into the state derivatives. In addition to the batch algorithm-like advantages, the minimum model error algorithm calculations are based upon sequential processing of the measurements, which, like the filter algorithms, greatly reduces the memory requirements and eliminates the need for large matrix manipulations. From the standpoint of algorithmic calculations, the minimum model error technique shares advantages of both batch and sequential estimation techniques.

Equations (18-23) describe a TPBVP for a given value of $W$. Once the TPBVP has been solved, the state estimates are substituted into the measurement model $g_k$ to produce the predicted measurements for the value of $W$. If the errors between the predicted measurements and the actual measurements have the same covariance as the prescribed measurement error covariance, then the state estimate is consistent. If the prescribed measurement error covariance is larger than the error covariance matrix between the predicted and actual measurements, then the predicted measurements are too close to the actual measurements. Too much model correction has been admitted. Thus, $W$ should be increased and the resulting TPBVP is solved for a new state estimate. If the prescribed measurement error covariance is smaller than the error covariance between the predicted and actual measurements, then the predicted measurements are not accurate enough. Too little model correction has been admitted, so $W$ should be decreased and the resulting TPBVP solved for a new state estimate.

Based on limited experience in applying this approach to a few example problems, a good starting value for $W$ is the inverse of the measurement error covariance matrix. This value is also intuitively reasonable, but the final value of $W$ is dependent on the model error, which is unknown a priori. The subsequent corrections to $W$ may be automated using standard search procedures. We note that choosing an appropriate value of $W$ is similar to "tuning" a Kalman filter, with the important distinction that the choice of process noise in the tuning of a Kalman filter is rarely based on more than the user's artistry. In minimum model error estimation, the covariance constraint determines a necessary condition upon the acceptable choice of $W$. At the present, the uniqueness of $W$ has not been resolved for multidimensional applications.

## Simple Scalar Example

To illustrate the application of the minimum model error approach, consider estimation of the state history of a scalar function of time for which noisy measurements are the only information available. No prior knowledge of the underlying dynamics is assumed. Thus, the system dynamic model equation is

$$\dot{\hat{x}}=0 \tag{24}$$

For simplicity, the measurements are direct measurements of the state itself, and the measurement noise is a zero mean Gaussian process with a presumed known variance of $\sigma^2$.

Using the minimum model error approach, the system model is modified by the addition of a to-be-estimated unmodeled effect as

$$\dot{\hat{x}}=0+d(t) \tag{25}$$

where $d(t)$ represents the dynamic model error. The measurements are given as

$$\tilde{y}_k=x(t_k)+v_k, \qquad k=1,2,\dots,M \tag{26}$$

where $\tilde{y}_k$ is the measurement at time $t_k$, $x(t_k)$ is the true state at time $t_k$, and $v_k$ is a zero-mean Gaussian sequence of presumed variance $\sigma^2$. The cost functional to be minimized [see Eq. (6)] is

$$J=\frac{1}{\sigma^2}\sum_{k=1}^{M}(\tilde{y}_k-\hat{x}(t_k))^2+\int_{t_0}^{t_f}d^2(\tau)Wd\tau \tag{27}$$

where $W$ is the to-be-determined weight on the integral sum-square model error term. The TPBVP which results from the minimization of $J$ with respect to $d(t)$ may be summarized as

$$\dot{\hat{x}}=d(t) \tag{28}$$

$$\dot{\lambda}=0 \tag{29}$$

$$d(t)=-\lambda/2W \tag{30}$$

$$\lambda(t_0^-)=\lambda(t_f^+)=0 \tag{31}$$

$$\lambda(t_k^+)=\lambda(t_k^-)+\frac{2}{\sigma^2}[\tilde{y}_k-\hat{x}(t_k)] \tag{32}$$

where $\lambda$ is the costate.

The algorithm proceeds according to the following steps:

1) Choose $W$.
2) Set $\hat{x}_0=x_0$.
3) Integrate forward to $t_f$, accounting for the jump discontinuities in $\lambda$ at each measurement time.
4) Check: Is $\lambda(t_f^+)=0$? If so, go to step 7.
5) Determine $\dfrac{\partial\lambda(t_f^+)}{\partial\hat{x}(t_0)}$, then $\Delta\hat{x}(t_0)$.
6) Adjust $\hat{x}_0$ by $\Delta\hat{x}(t_0)$; go to step 3.
7) Check the covariance constraint:

$$\frac{1}{M}\sum_{k=1}^{M}(\tilde{y}_k-\hat{x}(t_k))^2\approx\sigma^2?$$

8) If the covariance constraint is not satisfied, go to step 1.

The true state history for this example is taken as $x(t)=\cos(t)$. In Fig. 2, a set of 101 simulated measurements spanning the time interval $t_0=0$ to $t_f=10$ is shown. The measurements were simulated by adding a computer-generated Gaussian random sequence to the true state as

$$\tilde{x}_k=\cos(t_k)+v_k \tag{33}$$

The nominal variance of $v_k$ in Fig. 2 is 0.1, although the actual variance depends on the seed supplied to the random number generator. For the measurements shown, the actual error variance is 0.114, representing a typical error magnitude of approximately 0.34. The state itself has an average magnitude of 0.64, so that the typical measurement error in this example is more than 50% of the state (the signal to noise ratio is just under 2).

In Fig. 3, the minimum model error state estimate is shown along with the measurements and the true state history. Note that the state has been reconstructed to an error variance of 0.0085, considerably better than the measurement variance even in the total absence of a model. Note also that the model prediction variance (i.e., constant $\hat{x}=0$) is 0.717. Thus, the MME estimate error variance is 100 times smaller than the model error variance and 15 times smaller than the measurement error variance. The optimal estimate is significantly more accurate than either the model or the measurements, which is the implicit objective of state estimation algorithms.

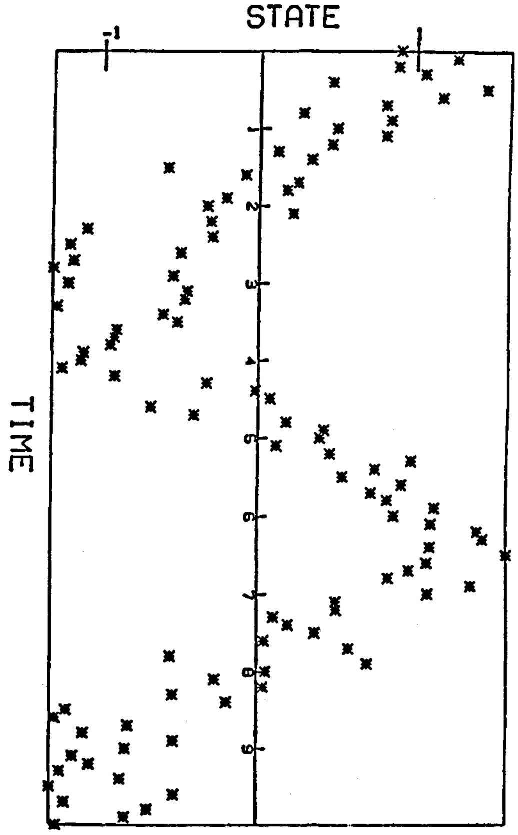


Fig. 2 Simulated measurements. Truth = cos($t$), measurement variance = 0.114.

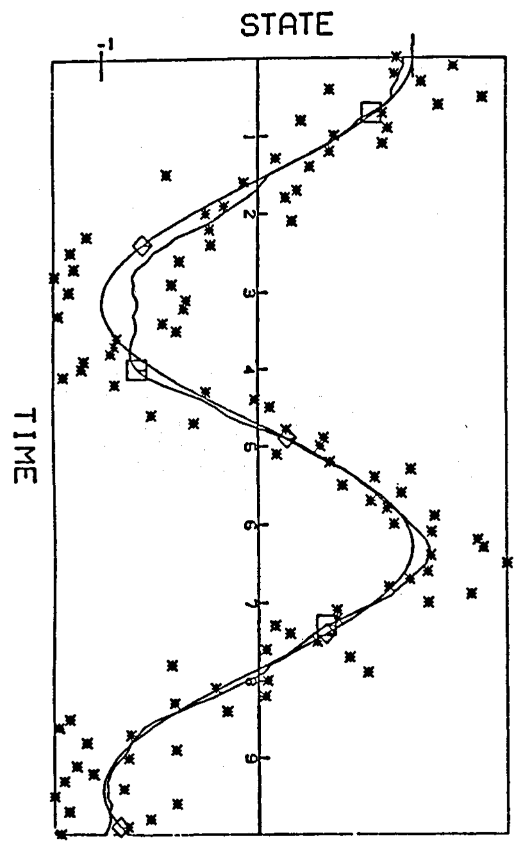


Fig. 3 *Measurements, ◇ truth, and □ state estimate. Measurement variance is 0.114, model variance is 0.717, estimate variance is 0.0085.

We note in passing that the usual filter strategies do not include mechanisms for correcting incorrect state dynamics models. Thus, the between-measurement filter predictions are based on integration of the original model. In the case of the present example, then, the filter estimate would consist of a discontinuous horizontal line, with jump discontinuities occurring at each measurement time.

## Conclusions

In this paper, a new optimal state estimation method has been developed for the postexperiment state estimation of discretely measured dynamic systems. The concept of a "covariance constraint" has been introduced as a necessary condition, marking a departure from the traditional criteria such as minimum variance or maximum likelihood estimation. The new method accounts for errors in the dynamic state model equations, and does not require or assume a priori knowledge of the dynamic model error. A minimum model error algorithm for obtaining state estimates which satisfy the covariance constraint is also derived. The method was demonstrated for a simple scalar problem, and the results indicate that the method is capable of obtaining very accurate state estimates in the presence of significant model error and significant measurement error.

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